Gravitational Collapse without a Remnant

**Zhe Chang · Cheng-Bo Guan · Chao-Guang Huang ·
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Abstract We investigate the gravitational collapse of a spherically symmetric, inhomogeneous star, which is described by a perfect fluid with heat flow and satisfies the equation of state $p = \rho/3$ or $p = C\rho^\gamma$ at its center. Different from the ordinary process of gravitational collapsing, the energy of the whole star is emitted into space. And the remaining spacetime is a Minkowski one at the end of the process.

Keywords Gravitational collapse · Inhomogeneous star · Energy mechanism for
gamma-ray bursts

1 Introduction

Gravitational collapse is one of the most important topics in general relativity. A large mount of models of gravitational collapse has been proposed [1–23]. In these models, the remnant of a gravitational collapse is possible a compact self-sustained star or a black hole, or a naked-singularity, depending on the initial condition of the collapse. A decade ago, in studying the critical phenomenon in gravitational collapse, first revealed by Choptuik [20, 21], Hirschmann and Eardley showed that the gravitational collapse of a complex scalar field may leave behind an approximately flat spacetime. Their model about the collapse, as a solution of Einstein field equations, possesses self-similarity [22]. Later, Schäfer and Goenner

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constructed a model of gravitational contraction of a radiating spherically symmetric body with heat flow [23], in which both initial mass and initial radius are infinitely large and all mass of the body can be radiated away eventually without forming an event horizon. Also about a decade ago, Fayos, Senovilla and Torres studied geometry matched by two spherically symmetric spacetimes through a timelike hypersurface from a very general point of view and exhausted all possible and qualitatively different matchings with their corresponding conformal diagrams for a flat Robertson-Walker model with a linear equation of state $p = \gamma \rho$ [24]. In particular, Fig. 9 and the time reversal of Fig. 11 in [24] show that a flat spacetime is left when a radiating homogeneous star (or Universe) with a linear equation of state radiates its all mass.

In the present paper, we propose a new approach to study the gravitational collapse of a spherical star. In our approach, a fluid star with finite initial mass and radius is supposed to be spherically symmetric, inhomogeneous, with the equation of state $p = \rho/3$ or $p = C\rho^\gamma$ at its center, and having heat flow outside it. Similar to the solutions given by Hirschmann and Eardley [22], Schäfer and Goenner [23] and Fayos et al. [24], all energy of the star in our model will be emitted in the process of collapse, and the remaining spacetime is an empty flat one. It is remarkable that for a star with about a solar mass and a solar radius, the energy at the order of 10^{54} erg will be emitted into space within about 8–600 s. As a result our model concludes that the inferred isotropic average luminosities are on the order of 10^{51-53} erg/s, which has the same order of magnitude for a gamma-ray burst.

The arrangement of the paper is as follows. The method of model construction and the equations governing the model are described in next section. The junction conditions and boundary conditions are listed in Sect. 3. In Sect. 4, as examples, several numerical solutions are presented. The concluding remarks are given in Sect. 5.

2 Formulation

In the pioneer paper to understand the late stages of stellar evolutions [1], a collapsing star is supposed to consist of homogeneous, spherically symmetric, pressure-less, perfect fluid and to be surrounded by an empty space. The interior of the star may be described by the Friedmann-Robertson-Walker metric [25]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right), \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on a unit 2-sphere, k is $\frac{8\pi G}{3}$ times of the initial energy density of dust in the unit of $c = 1$, and $a(t)$ is the solution of

$$\dot{a}^2(t) = k[a^{-1}(t) - 1].$$

In an astrophysical environment, a star usually emits radiation and throws out particles in the process of gravitational collapse. In this situation, the heat flow in the interior of a star should not be ignored and the exterior spacetime is no longer described by a Schwarzschild metric. To take the radiation of a star into account, the interior solution of the gravitational collapse of radiating stars should match to the exterior spacetime described by the Vaidya solution [26]

$$ds^2 = \left(1 - \frac{2GM(v)}{R}\right)dv^2 + 2dv dR - R^2 d\Omega^2, \tag{2}$$

which has been studied extensively [2–19, 24]. In particular, the gravitational collapse of a radiating spherical star with heat flow has been studied in an isotropic coordinate system [17–19]

$$ds^2 = dt^2 - B^2(t, r)(dr^2 + r^2 d\Omega^2), \tag{3}$$

where

$$B(t, r) = b^2(t)[1 - \lambda(t)r^2]^{-1}$$

with suitable $b(t)$ and $\lambda(t)$ functions. They called the solution the Friedmann-like solution.

We study the gravitational collapse of a spherically symmetric, inhomogeneous star in a proper-time reference, which can be written in a generalized Friedmann coordinate system

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - k(t)r^2} + r^2 d\Omega^2\right) \tag{4}$$

with k being a function of t.

The stress-energy tensor of the fluid with heat flow and without viscosity is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu, \tag{5}$$

where ρ and p are the proper energy density and pressure measured by the comoving observers respectively, u^μ is the 4-velocity of the fluid, and q^μ is the heat flow. u^μ and q^μ satisfy

$$u_\mu u^\mu = 1, \tag{6}$$

$$q_\mu u^\mu = 0. \tag{7}$$

For the spherically symmetric collapse, one has

$$u^\mu = (u^0, u^1, 0, 0), \tag{8}$$

$$q^\mu = (q^0, q^1, 0, 0). \tag{9}$$

The Einstein's field equations

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{10}$$

and the covariant conservation of stress-energy tensor give rise to

$$8\pi G\rho = \frac{k + \dot{a}^2 + 2a\ddot{a}}{a^2} + \frac{Y}{a^2 A^2}, \tag{11}$$

$$8\pi G p = -\frac{k + \dot{a}^2 + 2a\ddot{a}}{a^2} - \frac{X}{a^2 A^2}, \tag{12}$$

$$u_0^\pm = \sqrt{\frac{Y(X + Y) - \frac{Z^2}{2} \pm Z\sqrt{\frac{Z^2}{4} - XY}}{(X + Y)^2 - Z^2}}, \tag{13}$$

$$u_1 = aA\sqrt{u_0^2 - 1}, \tag{14}$$

$$8\pi G q_0 = \frac{1}{2a^2 A^2}\left[\frac{Y}{u_0} - (Y - X)u_0\right], \tag{15}$$

$$8\pi G q_1 = \frac{1}{2}\left[\frac{X}{u_1} - \frac{(Y - X)u_1}{a^2 A^2}\right], \tag{16}$$

with the identity

$$(8\pi G)^2 q_\mu q^\mu = -\frac{Z^2 - 4XY}{4a^4 A^4}, \tag{17}$$

where the over-dots denote the derivatives with respect to time t, and

$$A = \frac{1}{\sqrt{1 - k(t)r^2}}, \tag{18}$$

$$X = 3a\dot{a}A\dot{A} + a^2 A\ddot{A}, \tag{19}$$

$$Y = [2(k + \dot{a}^2 - a\ddot{a})A - a\dot{a}\dot{A} - a^2\ddot{A}]A, \tag{20}$$

$$Z = -\frac{4}{r}a\dot{A}. \tag{21}$$

X, Y, Z should satisfy the following solvable condition,

$$Z^2 \geq 4XY. \tag{22}$$

For spherical collapse, the dominant energy conditions [27] require

$$\rho - p \geq 0, \tag{23}$$

$$\rho + p \geq 2\sqrt{-q_\mu q^\mu}, \tag{24}$$

which lead to

$$X + Y > |Z| \geq 0. \tag{25}$$

3 Boundary Conditions

At the center of a star $r = 0$, $\dot{A} = \ddot{A} = 0$ and thus X, Z vanish. Hence,

$$u_0|_{r=0} = 1, \qquad u_1|_{r=0} = 0, \tag{26}$$

$$q_0|_{r=0} = 0, \qquad q_1|_{r=0} = 0, \tag{27}$$

$$8\pi G\rho|_{r=0} = 3a^{-2}(k + \dot{a}^2), \tag{28}$$

$$8\pi Gp|_{r=0} = -a^{-2}(k + \dot{a}^2 + 2a\ddot{a}). \tag{29}$$

The exterior solution outside the collapsing star is described by Vaidya metric (2). The interior solution and exterior solution should satisfy the Darmois junction conditions on the surface of the star [28, 29],

$$R_s = ar_s, \tag{30}$$

$$M(v) = \frac{a}{2G}(k + \dot{a}^2)r^3 \bigg|_s, \tag{31}$$

$$p_s^2 = |q_\mu q^\mu|_s, \tag{32}$$

and

$$[dt^2 - a^2 A^2 dr^2]_s = \left[\left(1 - \frac{2GM(v)}{R}\right)dv^2 + 2dv dR\right]_s, \tag{33}$$

where the subscript s denotes the value taken on the surface of the star. In particular, in (32) with the expressions (12), (17), (18), r should be replaced by $r_s(t)$, for (32) is only valid on the surface of the star. $M(v)$ is the Misner-Sharp mass [30, 31] of the star. (Appendix A gives the detailed discussion on the junction condition.) The equation of motion of the surface is

$$\frac{dr_s}{dt} = -a^{-2}A^{-2}\frac{u_1}{u_0}\bigg|_s. \tag{34}$$

In brief, with the help of the equation of state of fluid at the center of the star, (28), (29), (32) and (34) constitute a complete system of equations for $k(t)$, $a(t)$, $r_s(t)$. Once $k(t)$, $a(t)$ and $r_s(t)$ are obtained, one may directly get the expression for $\rho(r, t)$ and $p(r, t)$ from (11) and (12) and then determine the Misner-Sharp mass $M(v)$ and the surface radius R_s from (31) and (30).

4 Numerical Solutions

Without loss of generality, we choose $a(0) = 1$. In order to make numerical calculation conveniently, we introduce the dimensionless physical quantities as follows.

$$k \to kR_0^2, \qquad \dot{a} \to \dot{a}R_0, \qquad \ddot{a} \to \ddot{a}R_0^2, \qquad r \to r/R_0,$$
$$\dot{A} \to \dot{A}R_0, \qquad \ddot{A} \to \ddot{A}R_0^2, \qquad \rho \to \rho R_0^2/8\pi G,$$
$$p \to pR_0^2/8\pi G, \qquad q^\mu \to q^\mu R_0^2/8\pi G,$$

where R_0 is the initial radius of star. Note that we have already chosen the unit of $c = 1$.

In the following, we study several examples numerically.

4.1 $p = \rho/3$ at Center

Suppose that the equation of state at the center of a star take the radiation form, i.e. $p|_{r=0} = \rho/3|_{r=0}$, which was also discussed in Refs. [32, 33]. We are interested in the case that the surface of a star is almost stationary in the initial state, namely,

$$r_s(t = 0) = R_s(t = 0) = R_0, \tag{35}$$

$$\dot{R}_s(t = 0) \approx 0. \tag{36}$$

Therefore, we consider the initial conditions (35) and

$$\dot{a}(t = 0) = 0, \tag{37}$$

In such a case, only the u_0^- solution in (13) can reach (36) from the initial conditions (35) and (37). Figure 1 presents the numerical solution in the case. Since ρ and p are determined by (11) and (12), the initial state of the star is not homogeneous and the equation of state in the star is generally deviated from the radiation. In the figure, the time and radius r_s are in the units of R_0/c and R_0, respectively. In the process of evolution, $r_s(t)$ (dash curve) decreases monotonically while $a(t)$ (dotted curve) almost keeps a constant. $2M/R_s$ (real curve) in the unit of c^2/G increases first and then goes to 0 more quickly than R_s itself as $R_s \to 0$. It implies that the star will disappear without the appearance of a horizon in the process.

Figure 2 shows the evolution of ρ and p at the boundary, in unit of $c^4/(8\pi G R_0^2)$. The emission of the star arrives its maximum value at about $0.9R_0/c$ after the beginning of the collapse. In the late stage of the process, the equation of state at the boundary as well as at each point in the star tends to $p = \frac{1}{3}\rho$. At the end of the process both the energy density and the pressure become 0, which confirms that the whole star is radiated out into space in the process.

Furthermore, our numerical analysis shows that the star with $p|_{r=0} = \frac{1}{3}\rho|_{r=0}$ will radiate its whole mass in the process of the collapse, without the appearance of a horizon, for different initial values.

Fig. 1 Numerical solution of the gravitational collapse of spherical star with heat flow in the case $p|_{r=0} = \rho/3|_{r=0}$ and the u_0^- solution in (13). The horizontal axis is the time t in the unit of R_0/c. In the process of evolution, $r_s(t)$ (*dash curve*, in the unit of R_0) decreases monotonically while $a(t)$ (*dotted curve*) keeps almost a constant. $2M/R_s$ (*real curve*) in the unit of c^2/G increases first and then goes to 0 as $R_s \to 0$, which implies that the star disappears at the end of the collapse
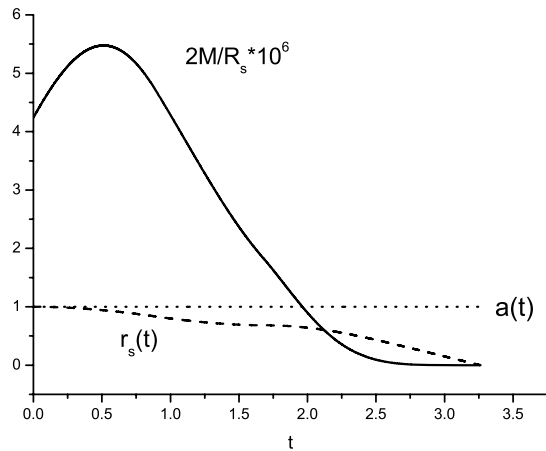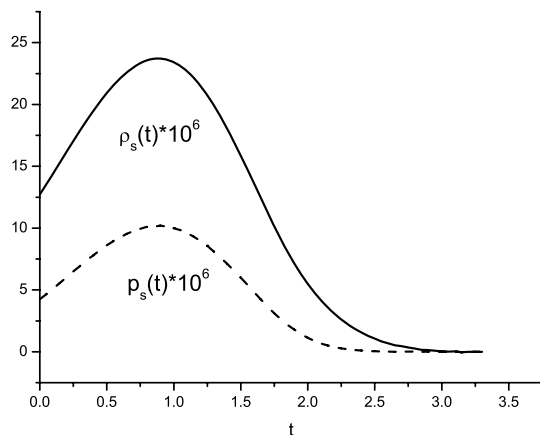

Fig. 2 Evolution of ρ and p at the boundary. The *horizontal axis* is the time in the unit of R_0/c. ρ and p are in the unit of $c^4/(8\pi G R_0^2)$

4.2 $p = C\rho^{\gamma}$ at Center

Now, let us suppose that the equation of state at the center have the forms of polytropes. Again, we only consider the u_0^- solution in (13) with the initial conditions (35) and (37). Figures 3 and 4 present the numerical solutions of Misner-Sharp mass in the cases $\gamma = 6/5$, $11/9$, $5/4$, $9/7$, $4/3$, $7/5$, $3/2$, $5/3$ and $C = 1$, respectively. The time is again in the unit of R_0/c. The behaviors of r_s for all cases are similar to the one in Fig. 1. These figures show that the evolutions of the stars with different adiabatic indexes have the similar behavior to that in Fig. 1 but with different collapsing time scale. Namely, the whole star will be emitted out without a remnant eventually in all these cases.

Figures 5 and 6 show the evolutions of ρ at the boundary for $p = \rho^{\gamma}$ with $\gamma = 9/7$, $5/4$, $6/5$, $11/9$, $5/3$, $3/2$, $4/3$, and $7/5$, respectively, in unit of $c^4/(8\pi G R_0^2)$. Figures 7 and 8 show the evolutions of p at the boundary for the same parameters. The times to emit all energy, T, are shown in Table 1. The last line in Table 1 gives the numerical values for the star with a solar radius initially. At the end of the process both the energy density and the pressure become 0, which confirms that the whole star is radiated out into space in the process.

Fig. 3 The ratios of mass to radius of collapsing stars for numerical solutions with $p|_{r=0} = \rho^{\gamma}|_{r=0}$, and $\gamma = 9/7, 5/4, 6/5, 11/9$. t is in the unit of R_0/c
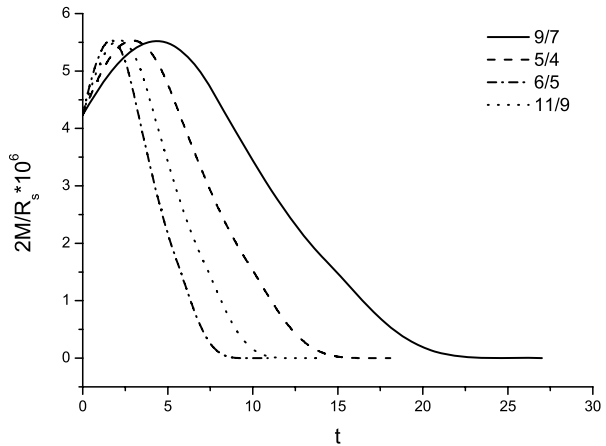

Fig. 4 The ratios of mass to radius of collapsing stars for numerical solutions with $p|_{r=0} = \rho^{\gamma}|_{r=0}$, and $\gamma = 5/3, 3/2, 4/3, 7/5$. t is in the unit of R_0/c
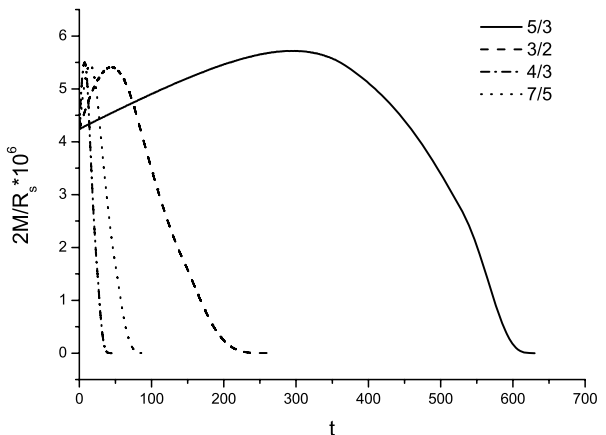
Fig. 5 Evolution of ρ at the boundary for the cases of $\gamma = 9/7$, 5/4, 6/5, and 11/9. The *horizontal axis* is the time in unit of R_0/c. ρ is in the unit of $c^4/(8\pi G R_0^2)$
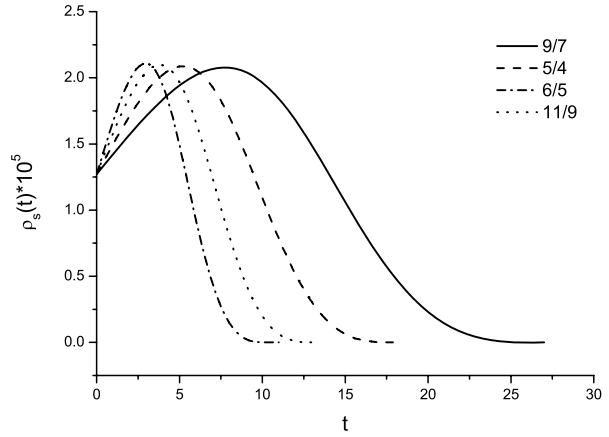

Fig. 6 Evolution of ρ at the boundary for the cases of $\gamma = 5/3$, 3/2, 4/3, and 7/5
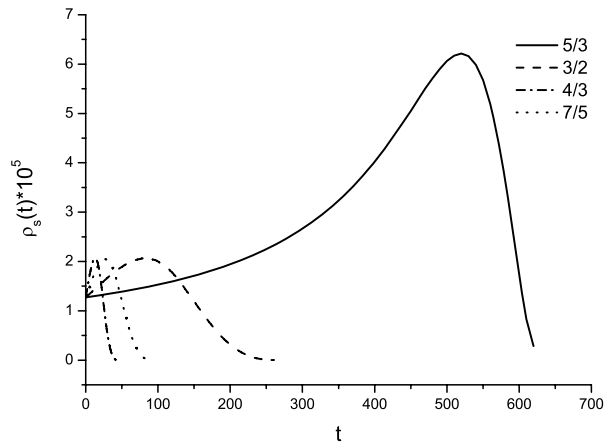

Fig. 7 Evolution of p at the boundary for the cases of $\gamma = 9/7$, 5/4, 6/5, and 11/9. The *horizontal axis* is the time in unit of R_0/c. p is in the unit of $c^4/(8\pi G R_0^2)$
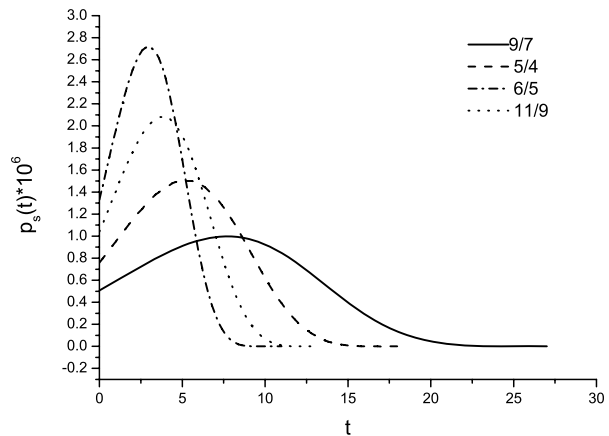
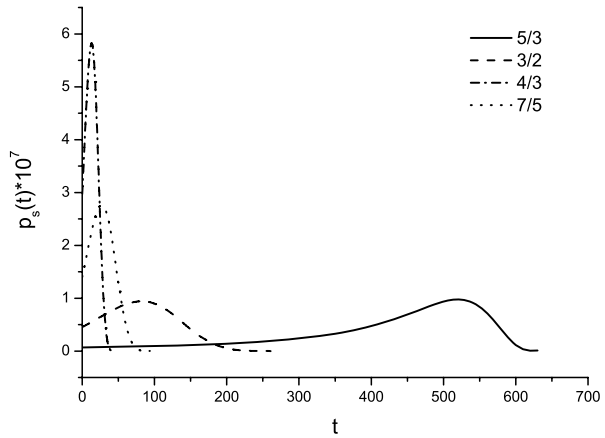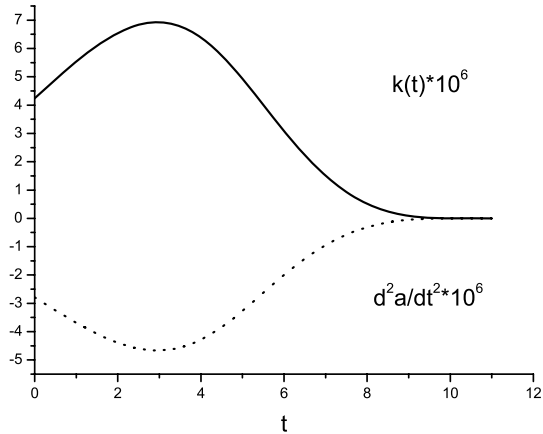**Fig. 8** Evolution of p at the boundary for the cases of $\gamma = 5/3$, $3/2$, $4/3$, and $7/5$

Table 1 Time to emit all energy for the star with $p = \rho^\gamma$ at its center

γ	6/5	11/9	5/4	9/7	4/3	7/5	3/2	5/3
cT/R_0	11	14	18	27	45	94	262	630
$T(s)$	25.3	32.7	42.0	62.7	104.5	218.7	608.2	1462.5

Fig. 9 Numerical solution for $k(t)$ and $\ddot{a}(t)$ in the case of $p|_{r=0} = \rho^{6/5}|_{r=0}$ and the u_0^- solution of (13)

Finally, we give the evolution of $k(t)$ and $\ddot{a}$ in Fig. 9. Obviously, $k(t)$ and $\ddot{a}$ have the same order of magnitudes, which is consistent with the fact that the equation of state at the center of the star serves as a free input parameter.

5 Concluding Remarks

We have shown a new approach to study the gravitational collapse of a spherical, inhomogeneous, fluid star with heat flow. By use of the Ansatz of the generalized Friedmann-Robertson-Walker metric, we obtain the new solutions of Einstein's field equations. These

solutions share the same properties: the initial mass and radius are finite and whole stars will be emitted in the process of the gravitational collapse in a finite interval, without the formation of a horizon, and a Minkowski spacetime is left at the end of the process. This is quite different from the standard evolution of the gravitational collapse of a star. In the standard process, a white dwarf, or a neutron star, or a black hole, or even a singularity will be formed at the final state and only a part of whole mass will be emitted. Obviously, the numerical solutions of the Einstein's field equations presented here are the description of another type of collapsing evolution beyond of the standard process.

As mentioned in Introduction, the numerical solutions with the property that all energy of the star in our model will be emitted in the process of collapse and an empty flat spacetime will be left behind have been obtained before [22–24]. Nevertheless, our solutions are more realistic than the previous ones. This is partly because in our solutions there is no self-similarity which will break down the asymptotic flatness of spacetime and partly because there is no initial singularity and the initial mass and radius are finite. Another reason is that the interiors of stars in nature are inhomogeneous and having the equation of state of polytrope. Our solutions presented here share these characters. In particular, when we apply our solution to a star with about a solar mass $M_\odot$, a solar radius $R_\odot$ and $p = \rho/3$ at the center, a huge mount of energy (about 1.8×10^{54} erg) will be emitted into space within 7.57 seconds, which are typical values for a gamma-ray burst [34–36]. If stars have the same size and $p = \rho^\gamma$ at their centers, the times to emit all energy, T, will range from 25 s to 1462.5 s. They are all reasonable values for gamma-ray bursts. And according to our numerical solutions we believe that if γ changes continuously, then the times to emit all energy will change continuously. Thus, the gravitational collapse of such a star might provide a new energy mechanism for gamma-ray bursts. Of course, detailed investigations of applying the solution to describe gamma-ray bursts are needed.

Appendix A: Boundary Conditions on the Star Surface

The motion of the star surface can be described by a time-like three-space Σ which divides the spacetime into the interior $\mathcal{V}^-$ and the exterior $\mathcal{V}^+$. The induced metric on and the extrinsic curvature of Σ are

$$ds^2_\pm|_\Sigma = (g^\pm_{\mu\nu} dx^\mu_\pm dx^\nu_\pm)_\Sigma, \tag{A.1}$$

$$K^\pm_{ij} = -n^\pm_\alpha \frac{\partial^2 x^\alpha_\pm}{\partial \xi^i \partial \xi^j} - n^\pm_\alpha \Gamma^{\pm\alpha}_{\mu\nu} \frac{\partial x^\mu_\pm}{\partial \xi^i} \frac{\partial x^\nu_\pm}{\partial \xi^j}, \tag{A.2}$$

respectively, where $x^\mu_\pm$, $g^\pm_{\mu\nu}$ and $\Gamma^{\pm\alpha}_{\mu\nu}$ are the coordinates, metrics and affine connections of $\mathcal{V}^\pm$ respectively; ξ^i is the intrinsic coordinate of Σ and $n^\pm_\alpha$ are the unit covariant vectors normal to Σ.

Following Israel [37], the junction conditions are the continuity of the induced metric and extrinsic curvature,

$$ds^2_-|_\Sigma = ds^2_+|_\Sigma, \tag{A.3}$$

$$K^-_{ij} = K^+_{ij}. \tag{A.4}$$

For the spherically symmetric collapse with heat flow and a radiating surface, a detailed discussion was given by Santos [29] with the interior solution in isotropic, comoving coordinates

$$ds_-^2 = A^2(\bar{t}, \bar{r})d\bar{t}^2 - B^2(\bar{t}, \bar{r})[d\bar{r}^2 + \bar{r}^2 d\Omega^2], \tag{A.5}$$

and the exterior Vaidya solution

$$ds_+^2 = \left(1 - \frac{2GM(v)}{R}\right)dv^2 + 2dvdR - R^2 d\Omega^2. \tag{A.6}$$

A similar discussion to Santos' can be taken for the more general interior solution in comoving coordinates,

$$ds_-^2 = e^{2\phi}d\bar{t}^2 - e^{2\psi}d\bar{r}^2 - \mathcal{R}^2 d\Omega^2, \tag{A.7}$$

where ϕ, ψ and $\mathcal{R}$ are functions of $\bar{t}$ and $\bar{r}$.

In the comoving coordinates, the equation of motion of the star surface is $\bar{r} = \bar{r}_\Sigma (=$ constant). The corresponding hypersurface equation is $f^-(\bar{t}, \bar{r}) = \bar{r} - \bar{r}_\Sigma = 0$. The spacelike vector $\partial f^-/\partial x_-^\alpha$ is normal to Σ and by normalization $g_-^{\mu\nu} n_\mu^- n_\nu^- = -1$ one will have

$$n_\alpha^- = (0, e^\psi, 0, 0). \tag{A.8}$$

By utilizing the metrics (A.6) and (A.7) in the first junction condition (A.3), one can have

$$R|_\Sigma = \mathcal{R}(\bar{t}, \bar{r}_\Sigma), \tag{A.9}$$

$$[(1 - 2GM/R)dv^2 + 2dvdR]_\Sigma = e^{2\phi}d\bar{t}^2|_\Sigma. \tag{A.10}$$

The equation of motion of star surface in the exterior space is given by the above equations. By supposing the hypersurface equation as $f^+(v, R) = R - R(v) = 0$, the normal covariant vector n_α^+ can be written as

$$n_\alpha^+ = \lambda\left(-\frac{dR}{dv}\bigg|_\Sigma, 1, 0, 0\right). \tag{A.11}$$

The normalization $g_+^{\mu\nu} n_\mu^+ n_\nu^+ = -1$ gives

$$\lambda^2[(1 - 2GM/R) + 2(dR/dv)]_\Sigma = 1. \tag{A.12}$$

Utilizing the equation (A.10), one easily gets $\lambda = e^{-\phi}\dot{v}|_\Sigma$, and then

$$n_\alpha^+ = e^{-\phi}\dot{v}(-dR/dv, 1, 0, 0)|_\Sigma = e^{-\phi}(-\dot{R}, \dot{v}, 0, 0)|_\Sigma, \tag{A.13}$$

where over-dots denotes the derivative with respect to $\bar{t}$.

The intrinsic coordinates of Σ can be conveniently chosen as $\xi^i = (\bar{t}, \theta, \varphi)$. Now with the obtained normal vectors and metrics, the induced extrinsic curvature can be calculated out. Their nonzero components are

$$K_{\bar{t}\bar{t}}^- = -e^{2\phi-\psi}\phi'|_\Sigma, \tag{A.14}$$

$$K_{\theta\theta}^- = e^{-\psi}\mathcal{R}\mathcal{R}'|_\Sigma, \tag{A.15}$$

$$K_{\varphi\varphi}^- = K_{\theta\theta}^- \sin^2\theta, \tag{A.16}$$

$$K_{\bar{t}\bar{t}}^{+} = e^{\phi}\left(\frac{\ddot{v}}{\dot{v}} - \dot{\phi} - \frac{GM}{R^2}\dot{v}\right)\bigg|_{\Sigma}, \tag{A.17}$$

$$K_{\theta\theta}^{+} = e^{-\phi}R\left[\dot{R} + \left(1 - \frac{2GM}{R}\right)\dot{v}\right]\bigg|_{\Sigma}, \tag{A.18}$$

$$K_{\varphi\varphi}^{+} = K_{\theta\theta}^{+}\sin^2\theta. \tag{A.19}$$

The second junction condition (A.4) (having only two independent equations because of spherical symmetry) can therefore be rewritten as

$$-e^{-\psi}\phi'|_{\Sigma} = e^{-\phi}\left(\frac{\ddot{v}}{\dot{v}} - \dot{\phi} - \frac{GM}{R^2}\dot{v}\right)\bigg|_{\Sigma}, \tag{A.20}$$

$$e^{-\psi}\mathcal{R}\mathcal{R}'|_{\Sigma} = e^{-\phi}R\left[\dot{R} + \left(1 - \frac{2GM}{R}\right)\dot{v}\right]\bigg|_{\Sigma}. \tag{A.21}$$

With the help of the metric continuity equations (A.9) and (A.10), the equation (A.21) can be reduced to

$$M(v) = \frac{\mathcal{R}}{2G}(1 + e^{-2\phi}\dot{\mathcal{R}}^2 - e^{-2\psi}\mathcal{R}'^2)\bigg|_{\Sigma} \equiv m(\bar{t}, \bar{r}), \tag{A.22}$$

where $m(\bar{t}, \bar{r})$ is the well-known Misner-Sharp mass function [30, 31]. Substituting $M(v)$ into the equation (A.21), one also gets

$$\dot{v}|_{\Sigma} = \frac{e^{\phi}}{e^{-\phi}\dot{\mathcal{R}} + e^{-\psi}\mathcal{R}'}\bigg|_{\Sigma}. \tag{A.23}$$

Substituting $M(v)$ and $\dot{v}$ into equation (A.20), one now gets

$$e^{-\phi-\psi}(\dot{\mathcal{R}}\phi' + \mathcal{R}'\dot{\psi} - \dot{\mathcal{R}}')|_{\Sigma}$$
$$= \left[e^{-2\phi}\left(\ddot{\mathcal{R}} + \frac{\dot{\mathcal{R}}^2}{2\mathcal{R}} - \dot{\mathcal{R}}\dot{\phi}\right) - e^{-2\psi}\left(\frac{\mathcal{R}'^2}{2\mathcal{R}} + \mathcal{R}'\phi'\right) + \frac{1}{2\mathcal{R}}\right]_{\Sigma}. \tag{A.24}$$

Suppose that the collapse material be a shear-free fluid with heat flow described by the stress-energy tensor (5). In the comoving coordinates, the unit four-velocity vector $u^{\mu} = (e^{-\phi}, 0, 0, 0)$ is normal to the space-like heat flow vector $q^{\mu} = (0, q^1, 0, 0)$. By solving the Einstein equation, one gets

$$8\pi Gq^1 = -e^{-\phi-2\psi}\frac{2}{\mathcal{R}}(\dot{\mathcal{R}}\phi' + \mathcal{R}'\dot{\psi} - \dot{\mathcal{R}}'), \tag{A.25}$$

$$8\pi Gp = e^{-2\phi}\frac{2}{\mathcal{R}}\left(\dot{\mathcal{R}}\dot{\phi} - \ddot{\mathcal{R}} - \frac{\dot{\mathcal{R}}^2}{\mathcal{R}^2}\right) + e^{-2\psi}\left(\frac{\mathcal{R}'^2}{\mathcal{R}^2} + \frac{2}{\mathcal{R}}\mathcal{R}'\phi'\right) - \frac{1}{\mathcal{R}^2}. \tag{A.26}$$

Now from the equations (A.24), (A.25) and (A.26) it is easy to see that

$$p|_{\Sigma} = \sqrt{|q^{\mu}q_{\mu}|}\bigg|_{\Sigma}. \tag{A.27}$$

In the end, one can conclude that for spherically symmetric collapse with heat flow but shear free, the boundary conditions are equations (A.1), (A.22) and (A.27), which are all invariant for the coordinate transformations on time-radial plane. Therefore although the coordinate is not comoving in our model, the above three boundary equations are still used directly.

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